Midterm Exam Calculus 2

9 January 2025, 18:30-20:30

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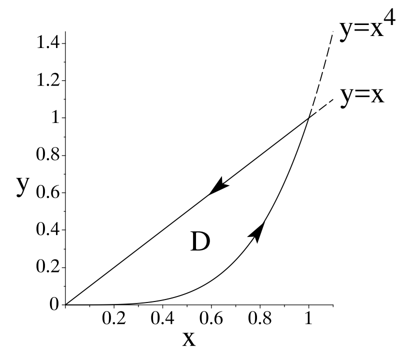
The midterm exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**5+5+10=20 Points**] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$
   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.
   (b) Let $\boldsymbol{u} = v\,\mathbf{i} + w\,\mathbf{j} \in \mathbb{R}^2$ be a unit vector, i.e. $v^2 + w^2 = 1$. Determine the directional derivative $D_{\boldsymbol{u}} f(0,0)$.
   (c) Use the definition of differentiability to determine whether f is differentiable at $(0,0)$.

2. [**2+15+8=25 Points**] Suppose the function $F : \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto F(u,v)$ is of class C^1 and is such that $F(-2,1) = 0$, $F_u(-2,1) = 7$ and $F_v(-2,1) = 5$. Let $G(x,y,z) = F(x^3 - 2y^2 + z^5, xy - x^2 z + 3)$.
   (a) Check that $G(-1,1,1) = 0$.
   (b) Show that we can solve the equation $G(x,y,z) = 0$ for z in terms of x and y (i.e., as $z = g(x,y)$ for (x,y) near $(-1,1)$ so that $g(-1,1) = 1$).
   (c) For the function g in part (b), compute the partial derivatives g_x and g_y at $(x,y) = (-1,1)$.

3. [**20 Points**] Heron's formula for the area A of a triangle whose sides have lengths x, y and z is
$$A = \sqrt{s(s-x)(s-y)(s-z)},$$
where $s = \frac{1}{2}(x+y+z)$ is the so-called *semiperimeter* of the triangle. Use the Method of Lagrange Multipliers to show that, for a fixed given perimeter p, the triangle with largest area is equilateral. (Hint: in computations it can be convenient to consider the squared area A^2 rather than A.)

4. [**25 Points**] Let D be the region in the first quadrant of $\mathbb{R}^2$ enclosed by $y = x^4$ and $y = x$ as shown in the figure on the right. For the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = xy\,\mathbf{i} + y^2\,\mathbf{j}$, show the following equality by computing both sides of the equation:
$$\iint_D \left(\frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \mathrm{d}A = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y,$$
where C is the piecewise smooth curve that forms the boundary of D with the orientation indicated by the arrows in the figure.



Solutions

1. (a) Using polar coordiantes $(x, y) = (r\cos\theta, r\sin\theta)$ we get for $r > 0$,

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r(\cos^3\theta + \sin^3\theta)$$

which goes to $0 = f(0,0)$ for $r \to 0$. Hence f is continuous at $(0,0)$.

(b) By definition

$$\begin{aligned} D_{\boldsymbol{u}} f(0,0) &= \lim_{t\to 0} \frac{f(tv, tw) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3v^3+t^3w^3}{t^2v^2+t^2w^2} - 0}{t} \\ &= \lim_{t\to 0} v^3 + w^3 \\ &= v^3 + w^3. \end{aligned}$$

(c) Choosing $\boldsymbol{u} = (1,0)$ in part (b) we get $f_x(0,0) = 1$ and similarly choosing $\boldsymbol{u} = (0,1)$ we get $f_y(0,0) = 1$. The linearization of f at $(0,0)$ hence is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x + y.$$

For the differentiability of f at $(0,0)$ we need to study the limit of

$$\frac{f(x,y) - L(x,y)}{\|(x,y)\|}$$

for $(x,y) \to (0,0)$. For $(x,y) \neq (0,0)$ we have

$$\frac{f(x,y) - L(x,y)}{\|(x,y)\|} = \frac{\frac{x^3+y^3}{x^2+y^2} - (x+y)}{(x^2+y^2)^{1/2}} = \frac{x^3+y^3-(x+y)(x^2+y^2)}{(x^2+y^2)^{3/2}}.$$

Using polar coordinates we get for $r > 0$,

$$\begin{aligned} \frac{x^3+y^3-(x+y)(x^2+y^2)}{(x^2+y^2)^{3/2}} &= \frac{r^3\cos^3\theta + r^3\sin^3\theta - (r\cos\theta + r\sin\theta)(r^2\cos^2\theta + r^2\sin^2\theta)}{(r^2\cos^2\theta + r^2\sin^2\theta)^{3/2}} \\ &= \cos^3\theta + \sin^3\theta - (\cos\theta + \sin\theta) \end{aligned}$$

which for example, for $\theta = 0$ gives 0 and for $\theta = \pi/4$, gives $\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} \neq 0$ and which hence no limit for $r \to 0$.
We conclude that f is not differentiable at $(0,0)$.

2. (a) To check that $G(-1,1,1) = 0$, note that substituting $x = -1$, $y = 1$, $z = 1$ in $G(x,y,z)$ gives $F((-1)^3 - 2(1)^2 + (1)^5, (-1)(1) - (-1)^2(1) + 3) = F(-1 - 2 + 1, -1 - 1 + 3) = F(-2, 1) = 0$. So $G(-1,1,1) = 0$.

(b) As F is of class C^1, G is of class C^1 too. If $\frac{\partial G}{\partial z}(-1,1,1) \neq 0$ then the Implicit Function Theorem gives that near the point $(x,y,z) = (-1,1,1)$ the level set $S = \{(x,y,z) \in \mathbb{R}^3 | G(x,y,z) = 0\}$ is locally a graph over the (x,y) plane, i.e. there exists a neighbourhood U of $(x,y) = (-1,1)$ in $\mathbb{R}^2$ and a neighbourhood V of $z = 0$ in $\mathbb{R}$ and a function $g : (x,y) \mapsto z = g(x,y)$ such that $g(-1,1) = 1$

and if $(x,y) \in U$ and $z \in V$ satisfy $G(x,y,z) = 0$ then $z = g(x,y)$.
At an arbitrary point $(x,y,z) \in \mathbb{R}^3$ we have

$$\begin{aligned}
\frac{\partial G}{\partial z}(x,y,z) &= \frac{\partial F(u(x,y,z),v(x,y,z))}{\partial z} \\
&= \frac{\partial F}{\partial u}(u,v)\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}(u,v)\frac{\partial v}{\partial z} \\
&= F_u(u,v)5z^4 + F_v(u,v)(-x^2).
\end{aligned}$$

Filling in $(u,v) = (-2,1)$ and $(x,y,z) = (-1,1,1)$ and using $F_u(-2,1) = 7$ and $F_v(-2,1) = 5$

$$\frac{\partial G}{\partial z}(-1,1,1) = 7 \cdot 5 \cdot 1^4 + 5 \cdot (-(-1)^2) = 30 \neq 0\,.$$

So we can apply the Implicit Function Theorem to prove the local existence of the function g.

(c) From the Implicit Function Theorem we get

$$g_x(-1,1) = -\frac{\frac{\partial G}{\partial x}(-1,1,1)}{\frac{\partial G}{\partial z}(-1,1,1)}$$

and

$$g_y(-1,1) = -\frac{\frac{\partial G}{\partial y}(-1,1,1)}{\frac{\partial G}{\partial z}(-1,1,1)}.$$

Similarly to part (b) we have

$$\begin{aligned}
\frac{\partial G}{\partial x}(x,y,z) &= \frac{\partial F(u(x,y,z),v(x,y,z))}{\partial x} \\
&= \frac{\partial F}{\partial u}(u,v)\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}(u,v)\frac{\partial v}{\partial x} \\
&= F_u(u,v)3x^2 + F_v(u,v)(y - 2xz).
\end{aligned}$$

Filling in $(u,v) = (-2,1)$ and $(x,y,z) = (-1,1,1)$ and using $F_u(-2,1) = 7$ and $F_v(-2,1) = 5$

$$\frac{\partial G}{\partial x}(-1,1,1) = 7 \cdot 3 \cdot (-1)^2 + 5 \cdot (1 - 2 \cdot (-1) \cdot 1) = 21 + 15 = 36\,.$$

giving

$$g_x(-1,1) = -\frac{36}{30} = -\frac{6}{5}.$$

Similarly

$$\begin{aligned}
\frac{\partial G}{\partial y}(x,y,z) &= \frac{\partial F(u(x,y,z),v(x,y,z))}{\partial y} \\
&= \frac{\partial F}{\partial u}(u,v)\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}(u,v)\frac{\partial v}{\partial y} \\
&= F_u(u,v)(-4y) + F_v(u,v)x.
\end{aligned}$$

Filling in $(u,v)=(-2,1)$ and $(x,y,z)=(-1,1,1)$ and using $F_u(-2,1)=7$ and $F_v(-2,1)=5$

$$\frac{\partial G}{\partial y}(-1,1,1)=7\cdot(-4)+5\cdot(-1)=-28-5=-33\,.$$

giving

$$g_y(-1,1)=\frac{33}{30}=\frac{11}{10}.$$

3. We use the method of Lagrange Multipliers to determine the largest area of the triangle. The function $g(x,y,z)=x+y+z$ gives the perimeter of a triangle with side length x, y and z. The constraint is hence given by $g(x,y,z)=p$ for a positive constant p. It is convenient to find the extrema of $A^2=s(s-x)(s-y)(s-z)$ rather than A under the constraint. Formally this is justified by the fact that $a\mapsto\sqrt{a}$ is a continuous, positive, increasing function on $[0,\infty)$.

   Let $f(x,y,z)=s(s-x)(s-y)(s-z)$ where the semiperimeter s is given by $s=\frac{1}{2}(x+y+z)=\frac{p}{2}$. Applying the method of Lagrange Multipliers gives $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ with $\lambda\in\mathbb{R}$ and the constraint $g(x,y,z)=p$, i.e.

$$\begin{aligned} f_x(x,y,z) &= \lambda g_x(x,y,z)\\ f_y(x,y,z) &= \lambda g_y(x,y,z)\\ f_z(x,y,z) &= \lambda g_z(x,y,z)\\ x+y+z &= p \end{aligned}$$

   which is equivalent to

$$\begin{aligned} -s(s-y)(s-z) &= \lambda && (1)\\ -s(s-x)(s-z) &= \lambda && (2)\\ -s(s-x)(s-y) &= \lambda && (3)\\ x+y+z &= p && (4) \end{aligned}$$

   Combining Equations (1) and (2) we see that $-s(s-y)(s-z)=-s(s-x)(s-z)$, so $x=y$. Combining (2) and (3) gives $y=z$, and hence $x=y=z$. (Looking at Equations (1) and (3) would also give $x=z$.) Together with the constraint $x+y+z=p$ we get $x=y=z=\frac{p}{3}$, where p is the perimeter of the triangle. So for a fixed perimeter p, the triangle with largest area is equilateral (all sides of the triangle have the same length).

4. (a) We start with the computation of the left hand side. We have

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0-x=-x.$$

   Hence

$$\begin{aligned} \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}A &= \int_0^1\int_{x^4}^{x}-x\,\mathrm{d}y\,\mathrm{d}x=\int_0^1-x(x-x^4)\,\mathrm{d}x\\ &= \int_0^1(-x^2+x^5)\,\mathrm{d}x=-\frac{1}{3}x^3+\frac{1}{6}x^6\Big|_0^1\\ &= -\frac{1}{3}+\frac{1}{6}=-\frac{1}{6}. \end{aligned}$$

We now compute the right hand side of the equation. We have $C = C_1 \cup C_2$ where C_1 corresponds to the part of the bondary where $y = x^4$ which has parametrization $\mathbf{r}_1(t) = (t, t^4)$, $0 \leq t \leq 1$. The tangent vector corresponding to the parametrization $\mathbf{r}_1$ gives the desired orientation shown in figure. The part C_2 corresponds to the part of the boundary where $y = x$ which can be parametrized by $\mathbf{r}_2(t) = \big(1-t, 1-t\big)$ with $0 \leq t \leq 1$. The tangent vector associated with $\mathbf{r}_2$ gives the desired orientation on C_2 shown in the figure. Using $\mathbf{r}_1'(t) = (1, 4t^3)$ and $\mathbf{r}_2'(t) = (-1, -1)$, we get

$$\begin{aligned}
\oint_C P\mathrm{d}x + Q\mathrm{d}y &= \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,\mathrm{d}t + \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\,\mathrm{d}t \\
&= \int_0^1 \big(t \cdot t^4, (t^4)^2\big) \cdot (1, 4t^3)\,\mathrm{d}t + \int_0^1 \big((1-t)(1-t), (1-t)^2\big) \cdot (-1,-1)\,\mathrm{d}t \\
&= \int_0^1 (t^5 + 4t^{11})\,\mathrm{d}t + \int_0^1 \big(-2(1-t)^2\big)\,\mathrm{d}t \\
&= \frac{1}{6} + \frac{4}{12} + \big(\frac{2}{3}(1-t)^3\big)\Big|_0^1 \\
&= \frac{1}{2} - \frac{2}{3} \\
&= -\frac{1}{6}
\end{aligned}$$

which agrees with the left hand side.